BULLETIN

OF THE

Society for the Promotion of Engineering Education

Vol. I LANCASTER, PA., MARCH, 1911 No. 8

EDITORIAL.

The Convention.—The nineteenth annual meeting of the Society will be held at Pittsburgh on Tuesday, Wednesday and Thursday, June 26, 27 and 28. At Madison the members voted unanimously to approve the invitation of the Carnegie Technical Schools to make the headquarters of the 1911 meeting at that institution. Pittsburgh presents many attractions to observant engineers and teachers on account of the concentration of manufacturing in the vicinity. The city is also well located from the standpoint of the distribution of the Society membership.

The meetings of the Society are increasingly valued as opportunities for becoming acquainted and maintaining acquaintance with fellow-workers who would be mere names but for meetings of this kind. The younger men need to know the older, and the latter need to keep in touch with the former, because as a rule the young men have the new and radical ideas. The radical ideas, however, require such modification as is indicated by experience. In view of the importance of the social side of the Society's work more than usual attention will be paid to it this year. This feature will be appreciated by the older members who have been attending the annual meetings regularly for many years. When one of these was asked, last year, if he had enjoyed the convention, he replied that it was fine. He stated, farther, that he had

been so busy meeting old friends that he had not been able to attend the meetings very much.

Notable Articles on Technical Education.—In a note to the editor printed in this issue Professor Clifton suggests that it would be well to print in the publications of the Society brief notes regarding articles of permanent value. He appends a few samples of what might be done, from the index of the *Electric Journal,* of Pittsburgh. If members will call attention to such articles, the titles with very brief abstracts will be promptly printed.

The Provisional Subscription for the Bulletin.—Last month the secretary sent out blank cards calling for provisional subscriptions for the BULLETIN for next year. The purpose of these cards was explained on pp. 395 and 396 of the February BULLETIN. As some members seem to have misunderstood the purpose of these cards a word of further explanation will be made.

Last year it was proposed to divide the membership fee into two parts for the purpose of complying with the postal law. On account of the experimental character of the BULLETIN and to avoid the complication of the divided fees the Council decided not to take any action in the matter but to determine by experiment what is necessary to secure admission of the BULLETIN to second-class rates. During the year the matter has been vigorously pushed with the result that we now know that we can not secure second-class rates without complying with the regulations of the post-office department as explained in the February BULLETIN.

Before the Council can act upon this matter at the coming meeting, it is necessary that they know the sentiment of the membership regarding the suggested change in the membership fee. Members who do not care for the BULLETIN will, by this plan, have a lower fee and they will get all that they did before with the larger fee. Members who value the BULLETIN enough to pay fifty cents extra for it are requested to turn in the subscription cards immediately so that a report can be made to the Council well before the meeting.

NEW MEMBERS.

The Council by letter ballot has elected to membership the 10 applicants whose names are listed on pages 397 and 398 of the February BULLETIN, and also the following. A list of all new members whose names do not appear in the January BULLETIN is enclosed loose with this number. This list should be pasted on page 389 of the January BULLETIN.

DURKEE, P. W., Professor of Engineering, Acadia University, Wolfville, Novia Scotia..... 1911

FREAR, J. B., Assistant Instructor in Farm Mechanics, University of Minnesota, State Farm, St. Paul, Minn..... 1911

KIRCHER, F. J., Instructor in Mechanical Engineering, University of Minnesota, Minneapolis, Minn..... 1911

MONROE, W. K., Engineer in charge of Estimating, The Brown Hoisting Machinery Co., Cleveland, O..... 1911

PARKS, C. W., Civil Engineer, United States Navy, Inspecting Engineer, General Electric Works, Schenectady, N. Y. 1911

RANDALL, J. A., Instructor in Mechanics and Heating, Pratt Institute, Brooklyn, N. Y..... 1911

SLICHTER, W. I., Professor of Electrical Engineering, Columbia University, New York City..... 1911

MEMBERSHIP LIST CORRECTIONS.

Since the publication of the January BULLETIN the following changes have been brought to the attention of the secretary.

Ayars, W. S. (1503 Hugo Place, Seattle, Wash.); Brown, J. S. (should be Brown, J. Stanford); Daniels, L. N. (East 6th St., Fowler, Ind.); Hormell, W. G. (resigned); Iyer, R. P. (Overseer Public Works Department, Parlakemidi, Ganjam District, India, year of joining, 1911); Jenson, Joseph (Springfield, Utah); Lewis, T. K. (resigned); Mark, R. M. (Rodman, New York State Barge Canal, 82 E. 7th St., Oswego, N. Y.); Taylor, T. U. (add to title Dean of Engineering Department); Woodward, C. M. (title is now Professor Emeritus of Mathematics and Applied Mechanics).

In the table on p. 373, January BULLETIN, there is confusion under the head, "Texas." The following is the correct distribution of membership between the two colleges: AUSTIN: (Univ. of Texas) A. C. Scott; T. U. Taylor; (2). COLLEGE STATION: (Texas Ag. Coll.) H. Y. Benedict; F. C. Bolton; E. J. Fermier; F. E. Giesecke; J. W. Kidd; J. C. Nagle; R. J. Potts; Chas. Puryear; (8).

In the geographical table the alphabetical order of the states under *M* and *N* is confused, but under each state the distribution is presumably correct.

LETTERS TO THE EDITOR.

EDITOR S. P. E. E. BULLETIN—

At the Madison Convention, in June, 1910, a plan was briefly outlined of a way for saving students and teachers time in graphical subjects. The plan was believed to be an improvement upon the common method in vogue in many colleges of mimeographing or hectographing the fundamental data of graphic problems. We are all familiar with the mussiness of either reproducing process and the time consumed by the teaching staff in preparing and printing material, even though it does save the student's time. The plan proposed at the convention was to have the fundamental data engraved and printed by some central agency, available to any college and supplied with problems designed by each. The plan has just had a term's trial at the Michigan Agricultural College in a course in descriptive geometry. The layouts for the problems were engraved at private expense and a near-by store was the distributing agent. No trouble arose from preparing data before class periods and if there had been danger, the distribution could have been affected in another way. The plan was eminently successful and will be continued. The exact percentage of time saved, or in other words the percentage of increase in work accomplished by the class in a given amount of time, has not been carefully worked out but it has been found to be considerable, enough to warrant a continuance of the plan. If teachers were to get together, exchange their problems in almost any graphical subject it might be possible to put before each other a vast amount of material, more than any one person could collect alone; and make the preparation of the material for a given class a very much simpler matter than it now is with its arduous individual work. The time seems ripe for this step in the evolution of educational methods.

Very truly yours,

EAST LANSING, MICH. VICTOR T. WILSON.

EDITOR S. P. E. E. BULLETIN—

Continuing our correspondence regarding the publication of notes on current articles treating of technical education, I would call your attention to the topical index of the *Electric Journal* of Pittsburgh. It is the custom of this magazine to publish at the end of each year a complete topical index covering the entire series to date. One of the subjects is "education." The enclosed extract shows the excellent method used in making up this index. The index covers only articles appearing in the *Electric Journal*. Would not some such plan as this be useful to members of the Society?

Sincerely yours,

H. T. CLIFTON.

PASADENA, CAL.

Commercial Research—C. E. Skinner. Investigation of properties of materials, processes, designs; development of new apparatus; critical study of existing designs; causes of failures; method of work, application of results, records. W-7400, Vol. V., p. 185, Apr., '08.

(E) Scientific Aids to Industrial Work—Chas. F. Scott. W-650, p. 182.

Engineering Responsibility—Chas. B. Dudley. An inquiry as to causes of failure and methods of improvement. Bad material; bad workmanship; bad design; unfair treatment. W-3800. Vol. VI., p. 483, Aug., '09.

(E) C. E. Skinner. A subject of vital importance to all. W-275, p. 452.

Education, Technical—(E) Comparison of President Humphrey's views with those of Mr. L. A. Osborne, expressed in an address before the A. I. E. E. W-800. Vol. I., p. 371, July, '04.

Education, Various Kinds of—Walter C. Kerr. Address at dinner of Cornell Alumni, Chicago, '05. W-1800. Vol. II., p. 289, May, '05.

Engineering and the College Graduate—H. W. Buck. The real benefits of college. Status of the engineer in society. W-1000. Vol. II., p. 685, Nov., '05.

Twentieth Century Engineer—Chas. F. Scott. W-2025. An address before the Engineers' Club of Philadelphia. Vol. IV., p. 222, April, '07.

(E) Chas. F. Scott. W-550, p. 184.

Getting On, Some Difficulties in—James Swinburne. Abstract of an address delivered to students of the British Institute of Electrical Engineers, Nov., '04. See (E) by Chas. F. Scott, p. 192. W-2600. Vol. II., p. 174, Mch., '05.

TREASURER'S STATEMENT.

The financial condition of the Society for the Promotion of Engineering Education is as follows on the eleventh day of February, 1911:

We have about $1,500.00 balance. One hundred and nineteen (119) members have paid since the second notice was sent out, representing about $455.00, *leaving about $1,000.00 still due.* We have paid all our bills received and the office expenses to date. We owe for the January BULLETIN, as we have not received a bill for that.

W. O. W.

Supplementing the above the secretary would state that the BULLETIN will cost, from January to June, about $900, of which somewhat over $500 may be returned from advertising. The PROCEEDINGS will cost, including postage, etc., about $1,000. Office expenses and printing for the balance of the year will require, say, $600.

COLLEGE NEWS.

University of Florida.—A contract has been let for the erection of the first part of an engineering building. It is expected that additional parts of the building, in the shape of wings adjoining the first part, will be built in the future as needed; and the plans have been worked out with special reference to development of this kind. The first part provides for class rooms, drafting rooms, dynamo laboratory, steam laboratory, testing laboratory, and hydraulic laboratory, and is to cost $40,000, exclusive of equipment.

Massachusetts Institute of Technology.—Professor D. C. Jackson spent part of January and February in England in conference with the British postal authorities regarding the purchase of the property of the National Telephone Company by the government. Professor Jackson's firm has been retained as advisors in appraising the value of the property.

University of Nebraska.—On January 18 the new mechanical engineering laboratories were dedicated.

Throop Polytechnic Institute.—A brief course in computations is given at this institution. While those in charge feel that the fundamental principles of computation can be taught in the separate departments, they wish to make sure that the proper attention is given to this work. A separate two-hour course is, therefore, given to all engineering students in the first semester of their freshmen year. The course has now been given twice with very satisfactory results. The work includes approximate computations, percentage errors, short methods, and computation by means of logarithmic and trigonometric tables. Slide rules, and tables and formulas from engineering hand-books are freely used. Especial attention is given to the checking of problems.

CONCERNING DRAFTING STANDARDS.

BY F. G. HIGBEE,

Professor of Drawing, The State University of Iowa.

A standard is any type, model, example, or authority with which comparison may be made. A standard in drafting is the base which all draftsmen should use as a measure or means of comparison. If these definitions are to hold, and an effort has been made to make them honest definitions, it will be at once seen that draftsmen today are laboring under about the same conditions as those which prevailed after the construction of the tower of Babel.

The slogan of the modern business world is standardization. In every organization, and in every business, of no matter what size or nature, progress and development are heralded by the slogan of standardization. The advantages of standardization are obvious and need no exploitation here, yet why is it that in the drafting room, the very place from which so many methods of standardization issue, real standards are lacking? Like music, drawing is a universal means of communication; it is the language of the designer. The forms by means of which musical ideas are expressed are standard but the forms by means of which constructive ideas are expressed are not. While drawing is a universal language it lacks in grammar and in uniformity of expression; it is a language of dialects.

The reason for this chaotic condition is plain. There is no absolute standard in drafting to which comparison can be made, and since there is no absolute standard there have grown up many arbitrary ones. Every draftsman has some sort of a standard to which he makes his work conform, but as many of the standards of this sort are the result of either

taste or accident they are apt to be unsound. The difficulty is not only to establish firmly *one sound standard* but also to abolish the *many unsound ones*. A sound standard, one which will prove acceptable to all, must be founded on real merit and good sense, and many of the arbitrary standards in use are unsound because these qualities are wholly lacking.

Many technical publications have printed articles on drafting-room practice, and many text-books on drawing have appeared in recent years, in which the subject of drafting standards is given scant consideration. Every writer has a standard, or an opinion, of his own and these are good, bad and indifferent, but no two of them agree. Writers seem to approach the subject with reluctance and usually skirt it nimbly, tossing off in passing one or two more of these usually worthless standards. When it is remembered that it is from such sources that the present drafting standards have originated it can be understood why the present chaotic conditions exist. Tradition, taste (or the lack of it), accident and coincidence, all seem to have played an important part in the formation of these standards, while too often does it appear that design, good sense, sober reflection, and economy of purpose are lacking.

By a comparison of the various standards of lines recommended by a number of authorities it will be seen that there is a wide variety of opinion and that, with a few exceptions, there is an entire lack of agreement as to what should constitute a standard. In the matter of the dimension line alone it will be seen that these authorities have many ways of expressing the same thing. Such looseness of expression is in no way tolerable. Such expressions as "ain't," "hain't," "he done it," "she seen it" are frowned upon in our spoken language, yet a slovenliness of expression every bit as evil is tolerated in the drafting language wherein exactness of expression is a prime requisite. If drafting is a universal language, and it should be if it is not, it should have universal conventions to express the same thing.

In all mechanical drawing only two kinds of lines are

needed to show the outlines of an object; visible lines and invisible lines. The visible lines should be strong unbroken black lines and the invisible lines should be dotted, or really very short dashed lines. Very little difference of opinion exists about the character of these most commonly used lines; it is in the other lines, the construction or auxiliary lines of the drawing, that there is great lack of uniformity. Of these lines the most important are center lines, dimension lines and extension lines. Of less importance are section lines and projection lines. Since these lines are usually superimposed upon the outlines of the drawing they must be of such weight and character that they will in no way be misunderstood and yet they must remain easy to make. The difficulty is not to find a set of standard lines to conform to these specifications, it is to get draftsmen to agree on which set conforms best.

Consider now the matter of standard hatchures for showing material in section. An investigation shows that nearly every authority has a standard of his own and that these standards agree with each other in but few instances. The following statement from Machinery's Reference Series explains the origin of many of these so-called standards and throws some refreshing side lights on their formation:

"There is no universally adopted standard for cross-sectioning for the purpose of indicating different materials. The chart above does not agree fully with the charts given in any of a number of text-books on mechanical drawing, but as these at the same time do not agree with one another, it has been assumed that the chart above may represent as good practice as some of those in the text-books. . . . There being no recognized standard, however, cross-sectioning alone should never be depended upon for indicating materials to be used. Written directions should also be given, or a small chart placed in the corner of the drawing indicating the conventions used in designating various materials."

The following extract from the widely-known text on "Machine Drawing," by Professor G. C. Anthony, is also illuminating: "Plate 17 illustrates the various types of section

lines available for the representation of different materials. Save in the case of cast-iron, Fig. 1, there is no general agreement among draftsmen as to the material each type of lining shall designate. Such a standard was proposed to the American Society of Mechanical Engineers (see *Transactions,* Vol. IX., page 107) but fortunately failed of adoption. It would only have added one more complication to the intricacies of modern machine drawing and made misunderstanding possible. If the slightest doubt as to the nature of the material represented is likely to occur, write the name of the material on the section. . . ."

To be quite candid, it would seem that were the situation less trying it would be profoundly comical. No standard for representing material in section exists which really does represent material in section; it is a case where too many standards have defeated the purpose for which each was created. A scrutiny of these many standards show that they lack merit and that there is no real reason why they should survive. Two of the prime requisites in section lining are lines of uniform weight and lines easy to make. In nearly every standard investigated these conditions were not observed. In many cases the lining is so difficult to make and the result so uncertain that the value of the representation is extremely doubtful. Either the practice of representing materials in section by hatchures should be abolished or a standard of section lining should be adopted which will render the practice less expensive and more certain.

In 1906 Professor George H. Follows published a text-book on mechanical drawing which he called a "Universal Dictionary of Mechanical Drawing." His effort is deserving of much credit for in this text he attempts to encourage uniformity in drawing by introducing a standard which his long experience led him to believe would be acceptable. Recently Professor John S. and David Reid undertook an investigation of the subject of drafting standards in a new way and published the results of their investigation in a revised edition of their text. These results were the subject of some sharp

criticism from Professor Follows and since these authorities seem to have the attainment of the same end in mind their disagreement is indeed unfortunate.

Professor Reid published as a standard the practice which, by means of a circular letter, he found was in use in the majority of drafting rooms of the two hundred leading engineering concerns which he consulted. Professor Follows attacked this standard on the ground that in most drafting rooms the practice is the result of accident and that a standard from such a source is, therefore, useless. While in general this charge may be true it would seem that, since only prominent concerns were consulted, any agreement among the majority of them upon a certain standard would indicate that it possessed some merit. Certainly any standard which has the endorsement of fifty or sixty leading engineering concerns behind it is worthy of serious consideration.

The problem of establishing drafting standards is entirely too difficult for an individual; such a movement must have the influence of a strong organization behind it. That greater uniformity in drawings is needed must be patent to all, and it must be equally patent that, in the face of this need, if national societies of engineers would make the effort, with the assistance and cooperation of the technical press, to formulate such a standard the movement would receive a widespread welcome and generous support. The matter of formulating the standard is not difficult of solution, the real difficulty lies in getting this solution universally adopted. It is for this reason that any movement to establish a standard is too great for an individual; the movement needs the support of influential and powerful organizations.

PROVISIONS FOR TESTING ROAD MATERIALS IN THE UNIVERSITY OF WASHINGTON.

BY C. E. MAGNUSSON,

Professor of Electrical Engineering, University of Washington.

During the present college year additions have been made to the laboratory equipment of the University of Washington for the complete examination of most of the important materials used in road and pavement construction. The laboratories referred to are: (1) the road laboratory in the Department of Civil Engineering for testing road metals and paving materials; (2) the laboratory of industrial chemistry, where the analyses of bituminous binders and similar materials are made.

Road Metals.—The principal work to be carried on in the road laboratory will be the examination and testing of road metals; *i. e.*, materials for broken stone pavements, and each student will make a complete examination of several typical samples from different parts of the State of Washington. While recognizing that in the great majority of cases, perhaps, the road builder has little opportunity for any choice in the selection of material, it is nevertheless true that (1) sometimes several materials are available, one of which will be more suitable than the others for the particular conditions of traffic and climate; (2) it would sometimes prove more economical in the end to import a more suitable material from a distance than to use an unsuitable local material; (3) standards of quality can be established only by comparison of service results with laboratory results. Light can best be thrown upon any of the above three points by laboratory examinations. Another point worthy of emphasis is that different samples of rock which answer to the same general geological description, and the appearance of which may be quite similar,

give widely different results when subjected to service tests. These differences may be anticipated by laboratory examination.

The road laboratory equipment covers the following tests: (1) Specific gravity, and absorption (in pounds of water per cubic foot of rock); (2) abrasion; (3) hardness; (4) toughness; (5) cementing power. The first test is not of great practical value since in the case of a rock of such porous structure as to be seriously weakened thereby, the pores are visible to the eye. Resistance to abrasion, grinding, and impact (in a degree depending upon the traffic to be withstood) are requisites for any satisfactory road metal. These qualities are determined by tests (2), (3) and (4). Cementing power, or the ability of the fine material to bind the larger pieces into a compact mass, is highly desirable under any conditions, for a rock with which no artificial binder is to be mixed.

The equipment for the abrasion test consists of the Deval machine, a series of cast iron cylinders mounted on a horizontal axis to which they are inclined at an angle of 30 degrees. The test is made by determining the loss, below $\frac{1}{16}''$, of 50 pieces of rock of standard size and totalling 5,000 gr., when placed in one of the cylinders and subjected to a fixed number of revolutions at standard speed. The hardness test is made by preparing, with a diamond drill, a rock core 25 mm. in diameter and three or four inches long. The end surfaces are squared off and the core is held (under a standard pressure) upon a revolving horizontal cast iron disc (the Dorry machine) upon which crushed quartz is fed. The loss in weight of the core is determined after 1,000 revolutions of the disc. The toughness test is made by preparing a cylinder, 25 mm. high, from the above mentioned rock core, and determining what height of blow, in centimetres, from a two kilogram hammer, is necessary to fracture it. The rock cylinders are prepared by first cutting the core into one inch pieces by a circular diamond saw, then finishing accurately to the

proper height by grinding upon a carborundum wheel, or a cast iron plate upon which emery is fed.

The cementation test is made by grinding small chips of rock with water in a ball mill, so as to form a stiff dough after a fixed number of revolutions of the mill. This dough is moulded into cylindrical briquettes, 25 mm. by 25 mm., under a standard pressure. After drying for twenty-four hours they are tested in a Page impact machine, which subjects them to a blow of constant height from a one-kilogram hammer.

The equipment for all tests conforms to the standards which have been adopted and used by the U. S. Office of Public Roads for a number of years. Several of these tests have been adopted as standards by the American Society for Testing Materials.

The laboratory will, when possible, make tests on rock samples for the State, and counties or towns in the State of Washington, without charge; in other cases a moderate fee is required. The work is in the immediate supervision of Julius Adler, instructor in the Department of Civil Engineering, and in charge of the work in highway engineering, under the general direction of Almon H. Fuller, dean of the College of Engineering.

Bituminous Binders.—In order to give students a knowledge of the most recent developments in highway construction, the laboratory of industrial chemistry has been equipped with aparatus to test all kinds of bituminous binders for roads and pavements. The regular work offered in the course, which began in February, comprises a series of lectures (for civil engineers taking the work in highway engineering) on the composition and properties of road binding materials, and is intended to serve as an interpretation and explanation of the laboratory tests. The latter work consists in the testing of commercial binders now on the market. In addition to the materials commonly used on the Pacific Coast, asfaltoil, asphaltoilene, Bermudez asphalt, crude oil, liquid asphalt, mineral rubber, paraffine flux, various grades of coal tar, paving pitch, and creosote, will be tested, thus familiarizing

the students with the properties of a large class of commercial materials.

Sufficient knowledge is probably not available to determine completely by laboratory tests the suitability of a bituminous binder for use under given conditions. Much progress has, however, been made in this connection. All of the tests now in use are also useful as recognition tests, and, as stated previously in connection with road metals, standards of quality can finally be established only by comparison of results obtained from actual use with those from laboratory tests.

Special apparatus procured for the work includes a Dow penetrating machine, a New York Testing Laboratory oven, an asphalt viscosimeter, briquette molds and asphalt pycnometers. The laboratory is equipped with air pressure, suction, gas, water, electric current, and a very complete stock of apparatus of a general nature. The tests to be made include specific gravity, per cent. of total bitumen, per cent. of free carbon (soot), penetration, viscosity and consistency, for both tars and asphaltic materials; flash point, per cent. of fixed carbon, and per cent. of loss by volatilization in the New York Laboratory oven, for asphaltic materials; examination by destructive distillation for tars. These tests include all of those most commonly employed in this country, making use of all standard methods that have been adopted by the American Society for Testing Materials.

The course is in charge of Dr. H. K. Benson, of the Chemistry Department, assisted by Mr. Adler, of the Department of Civil Engineering.

A SYLLABUS OF DIFFERENTIAL AND INTEGRAL CALCULUS.*

This syllabus is intended to include those facts and methods of the calculus which every student who has completed an elementary course in the subject should have so firmly fixed in his memory that he will never think of looking them up in a book. The topics here mentioned are therefore not by any means the only topics that should be included in a course of study, nor does the arrangement of these topics, as classified in the following table of contents, necessarily indicate the order in which they should be presented to a beginner.

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PART I. FUNCTIONS OF A SINGLE VARIABLE.

CHAPTER I. FUNCTIONS AND THEIR GRAPHICAL REPRESENTATION.

Function and argument. Tables. Graphs.

The elementary mathematical functions.

Continuity.

To find a mathematical function to represent an empirically given curve.

CHAPTER II. DIFFERENTIATION. RATE OF CHANGE OF A FUNCTION.

A. *Definitions and notation.* Rate of change of a function, or slope of the curve. Derivatives. Increments and differentials. Higher derivatives.

B. *To find the derivative when the function is given.*

Rules for differentiating the elementary functions. Differentiation of implicit functions, and of functions expressed in terms of a parameter.

C. *To find the derivative when the function itself is not given; setting up a differential equation.*

Useful theorems on infinitesimals.

* This syllabus forms part of a preliminary draft of the Report of the Committee on the Teaching of Mathematics to Students of Engineering, prepared by the Chairman, and here printed in advance in order to secure the widest criticism from all members of the Society. (See page 407.) *The Committee as a whole is not responsible for the report in its present form.* Criticisms and suggestions from any source are invited and may be addressed to the Chairman, Professor E. V. Huntington, 27 Everett Street, Cambridge, Mass.

D. *Applications of differentiation in studying the properties of a given function.* Slope; concavity; points of inflexion. Maxima and minima. Multiple roots. Small errors.

CHAPTER III. INTEGRATION AS ANTI-DIFFERENTIATION. SIMPLE DIFFERENTIAL EQUATIONS.

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Formal work in integration. Use of tables of integrals. Method of substitution, and method of integration by parts.

Simple differential equations.

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Fundamental theorem on the evaluation of a definite integral.

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Approximate methods of integration: squared paper; Simpson's Rule; the planimeter; expansion in series.

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CHAPTER V. APPLICATIONS TO ALGEBRA: EXPANSION IN SERIES; INDETERMINATE FORMS.

Taylor's Theorem with remainder. Maclaurin's Theorem. Important series.

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CHAPTER VI. APPLICATIONS TO GEOMETRY AND MECHANICS.

Tangent and normal. Subtangent and subnormal.

Differential of arc (in rectangular and polar coordinates).

Radius of curvature.

Velocity and acceleration in a plane curve.

PART II. FUNCTIONS OF TWO OR MORE VARIABLES.*

* In preparation.

CHAPTER I.

Functions and Their Graphical Representation.

1. Function and argument.—In many problems in practical life we have to deal with the relation between two variable quantities, one of which depends on the other for its value.

For example, the temperature of a fever patient depends on the time; the velocity acquired by a falling body depends on the distance fallen; the weight of an iron ball depends on its diameter, etc.

In general, if any quantity y depends on another quantity x, then y is called a *function* of x, written, for brevity, $y=f(x)$, and the independent variable x is called the *argument* of the function. More precisely stated, the notation $y=f(x)$ means that to every value of the argument x (within the range considered), there corresponds some definite value of the function, y; the value of y, or $f(x)$, corresponding to any particular value $x=a$ is denoted by $f(a)$.

If several values of y correspond to each value of x, we have what is called a "multiple valued function of x," which is really a collection of several distinct functions. For example, if $y^2=x$, then $y=\pm\sqrt{x}$, which is a double valued function of x.

Any mathematical expression involving a variable x is a function of x; but there are many important functional relations which cannot be expressed in any simple mathematical form.

2. A function is said to be *tabulated* when values of the argument (as many as we please, preferably at regular intervals) are set down in one column, and the corresponding values of the function are set down in another column, opposite the first. For example, in a table of sines, the angle is the argument, and the sine of the angle is the function.

3. A function may also be exhibited *graphically*, as follows: Lay off the values of the argument as abscissas along a (horizontal) axis, Ox, and at each point of the axis erect an ordi-

nate, y, whose length shall indicate the value of the function at that point; a curve drawn through the tops of these ordinates is called the *curve*, or the *graph*, of the function. It should be clearly understood, however, that it is the height of the ordinate up to the curve, rather than the curve itself, that represents the function.

In plotting the curve for any function, it is important to indicate on each axis the scale which is used on that axis, and the name of the unit. For example, if we plot distance as a function of the time, the units on the y-axis may represent feet, and those on the x-axis, seconds.*

The obvious method of obtaining the graph of the sum or difference of two functions directly from the graphs of those functions should be noted.

4. The elementary mathematical functions.—In many important cases, the relation between the function and the argument can be expressed by a simple mathematical formula. For example, if $s =$ the distance fallen from rest in the time t, then $s = \frac{1}{2}gt^2$. In such cases, the value of the function for any given value of the argument can be found by simple substitution in the formula.

The most important elementary mathematical functions are the following:

Algebraic functions: cx, c/x; x^2, x^3; $\sqrt{x}$ (x positive).

Here $\sqrt{x} =$ the positive value of y for which $y^2 = x$.

Trigonometric functions: $\sin x$, $\cos x$, $\tan x$ (x in radians).

Exponential function: e^x ($e = 2.718\ldots$).

Logarithmic function: $\log_e x$ (x positive).

The student should be thoroughly familiar with the curves of each of these functions, so as to be able to sketch them, or visualize them, at any moment; many of the essential properties of the functions can be obtained by inspection of the curve.

* It is not necessary that the lengths representing the units of x and y shall be equal; scales should be so chosen that the completed graph is of convenient size to fit the paper. In applications to geometry, however (see Chapter VI), the scales must be equal.

He should also be familiar with the formulas necessary for handling expressions involving these functions. The better drilled the student is in this formal algebraic work, the more rapid progress can he make in the really vital parts of the subject. (See chapters of this report on algebra and trigonometry.)

5. Next in importance are the following: the *hyperbolic functions,* which are coming more and more into use:

$$\sinh x = (e^x - e^{-x})/2, \quad \cosh x = (e^x + e^{-x})/2,$$
$$\tanh x = (e^x - e^{-x})/(e^x + e^{-x});$$

the *inverse trigonometric functions:*

$\sin^{-1} x$ = the angle between $-\pi/2$ and $+\pi/2$ radians (inclusive) whose sine is x;*

$\cos^{-1} x$ = the angle between 0 and π (inclusive) whose cosine is x;

$\tan^{-1} x$ = the angle between $-\pi/2$ and $+\pi/2$ (inclusive) whose tangent is x;

and the *inverse hyperbolic functions:*

$\sinh^{-1} x$ = the value of y for which $\sinh y = x$;

$\cosh^{-1} x$ = the positive value of y for which $\cosh y = x$;

$\tanh^{-1} x$ = the value of y for which $\tanh y = x$.

It should be noticed that the curves for the inverse functions can be obtained from the curves for the direct functions by rotating the plane through 180° about the line bisecting the first quadrant.

Formulas for the hyperbolic functions resemble those for the trigonometric functions, but the differences are so

* The symbol $\sin^{-1} x$ is often defined as simply "the angle whose sine is x"; but since there are many such angles, it is necessary to specify which one is to be taken as "the" angle, if the symbol is to have any definite meaning. Thus, if $\sin x = \frac{1}{2}$, x may equal $\pi/6$, or $5\pi/6$, etc.; but only one of these values, namely $\pi/6$, is properly denoted by the symbol $\sin^{-1} \frac{1}{2}$. Similarly for $\cos^{-1} x$ and $\tan^{-1} x$; and also for $\cosh^{-1} x$, which is like $\sqrt{x}$ in this respect. The conventions adopted to avoid ambiguity may be readily recalled from the figure, if we note that in each case the complete curve consists of two or more "branches," and that that one is taken as the "principal branch" which passes through the origin, or which lies nearest the origin on the positive side of the x-axis.

confusing that it is better not to try to memorize any formulas for the hyperbolic functions, but to look them up whenever they are needed. (The list in B. O. Peirce's Table of Integrals, for example, is entirely adequate.)

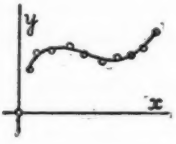
6. Continuity. A function $y=f(x)$ is said to be *continuous* at a given point $x=a$, if a small change in x produces only a small change in y; or more precisely, if $f(x)$ always approaches $f(a)$ as a limit when x approaches a in any manner.

A function may be *dis*continuous at a given point in two ways: (1) it may become infinite at that point, as $y=1/x$ at $x=0$; or (2) it may make a finite jump, as $y=\tan^{-1}(1/x)$ at $x=0$.* In each of these cases, the function is, properly speaking, not definite at the point in question.

A good example of a discontinuous function is the velocity of a shadow cast by a moving object on a zig-zag fence.

In what follows, we shall confine our attention to functions that are continuous, or that have only isolated points of discontinuity.

7. To find a mathematical function to represent an empirically given curve.—In many cases the form of the function is given only empirically; that is, the values of the function for certain special values of the argument are given by experiment, and the intermediate values are not accurately



known (for example, the temperature of a fever patient, taken every hour). In such cases, the methods of the calculus are not of much assistance, unless some simple mathematical law can be found which represents the function sufficiently accu-

* This function approaches $\pi/2$ when x approaches 0 from above, and $-\pi/2$ when x approaches 0 from below.

rately.* *This problem of finding a mathematical function whose graph shall pass through a series of empirically given points* is a very important one, which is much neglected in the current text-books. The complete discussion of the problem involves, it is true, the theory of least squares, which would undoubtedly be out of place in a first course in the calculus; but an elementary treatment of the problem in simple cases would be very desirable.†

The curves which are most likely to be worth trying, in any given case, are these:

$$y = a + bx \text{ (straight line)};$$
$$y = a + bx + cx^2 \text{ (parabola)};$$
$$y = a + c/(x + b) \text{ (hyperbola)};$$
$$y = a \sin (bx + c) \text{ (sine curve)}; \text{ and}$$
$$y = ax^m.$$

In testing this last curve, put $y' = \log y$, $x' = \log x$, and $a' = \log a$, and see whether y' and x' satisfy the straight line relation $y' = a' + mx'$; the use of "logarithmic squared paper" greatly facilitates the process.

The student should be familiar with all the possible forms of these curves, for various values of the constants a, b, c, and m.

* If no simple law can be found to represent the entire curve, it is sometimes possible to break up the curve into parts, and find a separate law for each part.

† Numerous examples may be found in John Perry's "Practical Mathematics," and in F. M. Saxelby's "Practical Mathematics" (Longmans, 1905).

CHAPTER II.

Differentiation. Rate of Change of a Function.

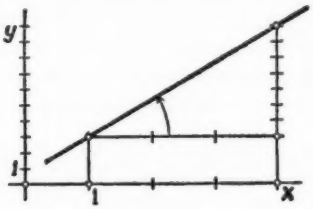
For the sake of clearness, this chapter is divided into four parts, A, B, C, D.

A. Definitions and Notation.

8. Rate of change of function; slope of curve.—Given a function, $y = f(x)$, one of the most important questions we can ask about it is, what is the *rate of change* of the function at a given instant?

For example, the distance of a railroad train from the starting point is a function of the time elapsed, and we may ask, what is the rate of change of this distance? the answer is, so-and-so many *miles per hour*. Again, the volume of a metal sphere is a function of the temperature, and we may ask, what is the rate of change of this volume? the answer is, so-and-so many *cubic inches per degree*.

If the graph of the function is a straight line, then clearly the rate of change of the function will be constant; for, at



any instant, (change in y)/(change in x) = the slope of the line.

If the scales along x and y are the same, the *slope of the line* = tan ϕ, where ϕ is the angle which the line makes with the x axis. If the scales are not the same, the slope of the line may still be interpreted as the ratio of the "side opposite" to the "side adjacent" in the triangle of reference for ϕ, provided each side is measured in the proper units. For example, in the figure, slope = 7/3.

If the graph is not a straight line, the meaning of "rate of change" at a given instant must be made more precise, as follows: Consider a particular value, $x = x_0$; give x an arbitrary change, Δx, and compute the corresponding change in y, namely, $\Delta y = f(x_0 + \Delta x) - f(x_0)$. Then the ratio $\Delta y/\Delta x$ may be called the AVERAGE *rate of change of the function during the interval from* $x = x_0$ *to* $x = x_0 + \Delta x$. (Geometrically, $\Delta y/\Delta x$ is the slope of the secant PQ in the figure.) Now let Δx approach zero, so that the interval in question closes down about the point $x = x_0$. Then *the ratio* $\Delta y/\Delta x$ *will in general approach a definite limit, and this limit is called the* ACTUAL *rate of change at the point* $x = x_0$. (Geometrically, the limit of $\Delta y/\Delta x$ is the slope of the tangent at P.*)

9. Derivatives. The *rate of change* of a function $y = f(x)$ at any point, or the *slope of the curve* at that point, is called the **derivative** of the function at that point, and is denoted by $f'(x)$, or $D_x y$, or y'.

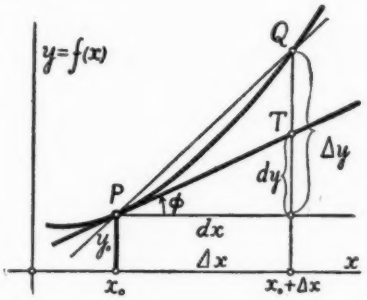
The notation $\dot{y}$ is also used, but only when the independent variable is the time.

This definition of the derivative of a function as the limit of $\Delta y/\Delta x$ is the fundamental concept of the differential calculus. It is desirable that the meaning of the definition be made perfectly clear, by numerous and varied illustrations, before any formal work in differentiation is taken up.

10. Increments and Differentials.—The value Δx is called the *increment* given to x, and Δy the corresponding *increment*

* The sense in which the tangent line is the "limit" of the secant lines should be made thoroughly clear. First, the tangent is a *fixed* line; secondly, the secant is a *variable* line, depending on the value given to Δx (that is, for every value of Δx, except the value 0, there is a corresponding position of the secant); thirdly, the angle between the tangent and the secant can be made to become *and remain* as small as we please by taking Δx sufficiently small. The tangent line itself does not in general belong to the series of secant lines; it is not in any sense the "last one" of the secants; it is a separate line, which bears a special relation to the series of secant lines, as described. The student may readily convince himself that the tangent is the *only* line through P that has the property just stated.

produced in y. The value which Δy *would have* if the curve coincided with its tangent (see figure) is called the **differential** *of* y and is denoted by dy.



In case of the independent variable x, the differential of x is, by definition, the same as the increment: $dx = \Delta x$.

The use of differentials gives us a *new notation for the derivative*,

$$f'(x) = \frac{dy}{dx}.$$

Both these notations are in common use.

Notice that Δy and dy are both variables which approach zero when we make Δx approach zero; dy/dx is a constant, equal to $\tan \phi$; $\Delta y/\Delta x$ is a variable, approaching $\tan \phi$ as a limit. Hence we may write:

$$\lim_{\Delta x = 0} \frac{\Delta y}{\Delta x} = f'(x) = D_x y = \frac{dy}{dx} = \tan \phi,$$

and

$$dy = f'(x)dx.$$

These relations between increments, differentials, and derivatives should be thoroughly mastered; they are readily recalled by the figure. Note especially that Δx and dx are quantities measured in the same unit as x; and Δy and dy in the same unit as y; while the derivative, dy/dx, that is, the slope, is (in general) measured in a compound unit (like miles per hour).

If the lengths representing the units of x and y are not equal, the slope of the curve, or $\tan \phi$, must be understood in the generalized sense explained above.

The process of finding the derivative, or the equivalent process of finding the differential of the function in terms of the differential of the argument, is called *differentiation.*

11. Higher derivatives. Since the slope of the curve varies, in general, from point to point, the derivative, $f'(x)$, is itself a function of x (often called the *derived* function); the derivative of $f'(x)$ is called the *second derivative* of the given function, and is denoted by $f''(x)$, or $D_x^2 y$, or y'' (or by $\ddot{y}$ in case the independent variable is the time); and so on for the higher derivatives.

It is also easy to define second, third, . . . *differentials,* but they are not of great importance. One matter of notation, however, should be carefully noticed, namely that d^2y/dx^2 is commonly used to denote $f''(x)$, that is $\dfrac{d(dy/dx)}{dx}$, and not, as one might expect $\dfrac{d(dy)}{(dx)^2}$.

As an example where the distinction is important, consider

$$x = \theta - \sin \theta \text{ and } y = 1 - \cos \theta,$$

where θ is the independent variable.

B. To Find the Derivative when the Function is Given.

12. Formal work in differentiation. The student should be thoroughly familiar with the results of differentiating all the elementary functions. A list of the formulas which should be memorized is given below; any other formulas should be worked out as needed, or looked up in a book.

To establish these formulas, first prove the following important limits:

$$\lim_{\Delta u \doteq 0} \frac{\sin \Delta u}{\Delta u} = 1, \quad \text{and} \quad \lim_{\Delta u \doteq 0} \frac{1 - \cos \Delta u}{\Delta u} = 0,$$

provided u is in radians; and

$$\lim_{n=\infty} \left(1 + \frac{1}{n}\right)^n = e = 2.718\cdots;*$$

anl hence prove the formulas for differentiating the *sine* and the *logarithm*.

The proofs of the other formulas present no difficulty.

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* These limits having been established, it can then be shown that

$$\lim_{\Delta u \doteq 0} \frac{\sin(u + \Delta u) - \sin u}{\Delta u} = \frac{\pi}{180} \cos u, \text{ if } u \text{ is measured in degrees,}$$

$$= \cos u, \text{ if } u \text{ is measured in radians;}$$

$$\lim_{\Delta u \doteq 0} \frac{\log(u + \Delta u) - \log u}{\Delta u} = (0.4343\cdots)\frac{1}{u}, \text{ if the base is 10,}$$

$$= \frac{1}{u}, \text{ if the base is } e = 2.718\cdots.$$

The reason for choosing the radian as the unit angle, and e as the base of the "natural" system of logarithms is the simplification in the formulas for the derivatives of the sine and the logarithm which results from this choice.

Rules for Differentiating the Elementary Functions of a Single Variable.*

(The first four of these rules are the fundamental ones, from which all the others can be derived.)

The differential of a constant is zero: —

$$dk = 0.$$

The differential of the LOGARITHM to the base e of any function is one over that function, times the differential of the function: —

$$d(\log_e x) = \frac{1}{x}\,dx \qquad (e = 2.718\ldots).$$

The differential of the SINE of any function (in radians) is the cosine of that function, times the differential of the function:

$$d(\sin x) = \cos x\,dx.$$

The differential of the sum [or difference] of two functions is the differential of the first plus [or minus] the differential of the second: —

$$d(u \pm v) = du \pm dv.$$

The differential of a constant times any function is the constant times the differential of the function: —

$$d(kx) = k\,dx.\dagger$$

The differential of a function to any constant power is the exponent of the power, times the function to the power one less, times the differential of the function: —

$$d(x^n) = nx^{n-1}dx.$$

Useful special cases of this rule are: —

$$d\sqrt{x} = \frac{1}{2\sqrt{x}}\,dx; \qquad d\left(\frac{1}{x}\right) = -\frac{1}{x^2}\,dx.$$

The differential of e with a variable exponent is e with the same exponent, times the differential of the exponent: —

$$d(e^x) = e^x\,dx \qquad (e = 2.718\ldots).$$

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* *All these rules remain valid when the word "derivative" is put in place of "differential," and the symbol "D" in place of "d."*

† *To prove this and the next five rules, let y = the function, and take the logarithm of both sides before differentiating.*

The differential of the product of two functions is the first times the differential of the second, plus the second times the differential of the first: —

$$d(uv) = u\,dv + v\,du.$$

The differential of the quotient of two functions is the denominator times the differential of the numerator, minus the numerator times the differential of the denominator, all divided by the denominator squared: —

$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$$

The differential of the cosine of any function is minus the sine of that function, times the differential of the function: —

$$d(\cos x) = -\sin x\,dx.*$$

The differential of the tangent of any function is the secant-square of that function, times the differential of the function: —

$$d(\tan x) = \sec^2 x\,dx.\dagger$$

The differentials of the inverse sine, the inverse cosine, and the inverse tangent, of any function, are given by the following formulas, which the student may put into words for himself: —

$$d(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}\,dx,\ddagger \qquad (-\tfrac{1}{2}\pi \leqq \sin^{-1} x \leqq \tfrac{1}{2}\pi)$$

$$d(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}\,dx, \qquad (0 \leqq \cos^{-1} x \leqq \pi)$$

$$d(\tan^{-1} x) = \frac{1}{1+x^2}\,dx. \qquad (-\tfrac{1}{2}\pi \leqq \tan^{-1} x \leqq \tfrac{1}{2}\pi)$$

[To find the differential of u to the vth power, where u and v are any functions, let

$$y = u^v,$$

and take the logarithm to base e of both sides before differentiating. — Similarly, to find the differential of the logarithm of u to any base v, let

$$y = \log_v u, \text{ whence } v^y = u;$$

then differentiate both sides.]

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* *Proof:* $\cos x = \sin(\frac{1}{2}\pi - x)$.

† *Proof:* $\tan x = \sin x / \cos x$.

‡ *Proof: Let* $y = \sin^{-1} x$, *that is,* $\sin y = x$; *then differentiate both sides. — Similarly for the next two formulas.*

The rules on these two pages suffice for the differentiation of any elementary function; they should be carefully memorized.

The differentials of the hyperbolic functions are given by the following formulas, which are also worth remembering:

$$d\sinh x = \cosh x\,dx; \quad d\cosh x = \sinh x\,dx;$$
$$d\tanh x = \operatorname{sech}^2 x\,dx;$$

hence,

$$d\sinh^{-1}x = \frac{dx}{\sqrt{x^2+1}}, \quad d\cosh^{-1}x = \frac{dx}{\sqrt{x^2-1}}, \quad d\tanh^{-1}x = \frac{dx}{1-x^2}.$$

13. Differentiation of implicit functions, and of functions expressed in terms of a parameter.

(*a*) Suppose we have an equation connecting x and y, but not giving y explicitly as a function of x; as, for example, $9x^2+4y^2=36$. In finding dy/dx in cases of this kind, instead of first solving the equation for y in terms of x, and then differentiating, it is usually better to *differentiate the equation through as it stands* (remembering that both x and y are variables); thus, in the present example we have $18x\,dx+8y\,dy=0$, whence, $dy/dx=-9x/4y$. This result can then, if desired, be expressed wholly in terms of x, by aid of the original equation.

(*b*) Again, suppose y is given as a function of u and v, where u and v are both functions of x; as, for example, $y=u^2+v\sin u$. Differentiating both sides by the regular rules, we have $dy=2u\,du+v\cos u\,du+\sin u\,dv$, whence, collecting the terms in du and dv, and dividing by dx,

$$\frac{dy}{dx} = (2u+v\cos u)\frac{du}{dx} + (\sin u)\frac{dv}{dx}.$$

This result shows how the rate of change of y depends on the rates of change of u and v, which are supposed to be known.

(*c*) Finally, both x and y may be given as functions of a third variable, t; as, $x=F(t)$, $y=f(t)$. To every value of this auxiliary variable, or "parameter," t, there corresponds a *pair* of values of x and y, so that here again y is indirectly determined as a function of x. Of course if we can eliminate t

we shall have a single equation connecting x and y; but it is often more convenient to *keep the equations in the parameter form.* Thus, to find dy/dx, we have merely to differentiate both of the given equations: $dx = F'(t)\,dt$, $dy = f'(t)\,dt$; and then divide one result by the other: $dy/dx = f'(t)/F'(t)$.

C. To Find the Derivative when the Function Itself is not Given; Setting up a Differential Equation.

14. In many cases it is required to find the rate of change of a function when the function itself is not directly given; in fact it is often easier to find the derivative of a function than it is to find the function itself.

For example, a hemispherical bowl of radius r, full of water, is being emptied through a hole in the bottom; find the rate of change of the volume of water drawn off, regarded as a function of the distance, y, between the level of the water and the center of the bowl. To compute this value directly from the definition, we notice first that the increment ΔV produced in V by an increment Δy given to y will have a value between $\pi(r^2 - y^2)\Delta y$ and $\pi[r^2 - (y + \Delta y)^2]\Delta y$; dividing either of these values by Δy, and taking the limit of the ratio $\Delta V/\Delta y$, we find at once $dV/dy = \pi\ (r^2 - y^2)$, which gives the required value of dV/dy for any value of y from $y = 0$ to $y = r$.

This process of finding the derivative directly from first principles, as the limit of the ratio of the increments, when the function itself is not given, is called "setting up a differential equation," since the result of the process is an equation between the differentials of the function and of the argument.*

Every problem of this kind is a problem in finding the *limit of the ratio of two variable quantities, each of which is approaching zero;* and in this connection the following theorems on infinitesimals are extremely useful, if not indispensable.

* The problem of finding the relation between the quantities themselves when the relation between their differentials is known will be discussed in the next chapter.

15. Theorems on infinitesimals.

Def. Any variable quantity that approaches 0 as a limit is called an *infinitesimal.* For example, Δx, Δy, dx, dy, are infinitesimals.

The erroneous notion that an infinitesimal is a *constant* quantity which is "smaller than any other quantity, however small, and yet not zero" should be carefully avoided.

Notation. The notation $\lim x = a$, or $x \to a$ (read: "x approaches a as a limit"), means that $x = a + \epsilon$, where ϵ is a variable approaching zero. Thus a statement expressed in terms of "lim" or "$\to$" can always be translated into an *equation*, which can then be handled by the ordinary rules of algebra. The symbol $\to$ is preferable to $=$ and seems likely to replace it.

Def. If α and β are infinitesimals, and $\lim (\alpha/\beta) = 0$, then α is said to be an *infinitesimal of higher order* than β.

For example, if $\Delta u = \epsilon \cdot \Delta v$, where ϵ itself approaches 0, then Δu is of higher order than Δv. Again, $1 - \cos \Delta\theta$ is of higher order than $\Delta\theta$.

If the difference between two infinitesimals is of higher order than either, then their ratio approaches 1 as a limit; and conversely, if the ratio of two infinitesimals approaches 1, then their difference is of higher order than either. Two infinitesimals having this relation may be called "**similar**" or "**equivalent**" infinitesimals.

Important examples are the following: a convex arc of a curve, and the chord of that arc, are "similar" infinitesimals. Again, $\sin \Delta x$ and $\tan \Delta x$ are both "similar" to Δx, provided Δx is in radians.

First Replacement Theorem for Infinitesimals. *In finding the limit of the ratio of two infinitesimals, either of them may be replaced by a "similar" infinitesimal, without affecting the value of the limit.*

As explained above, two infinitesimals are "*similar*": (1) if the *difference between them is of higher order than either;* or (2) if the *limit of their ratio is 1.* (Sometimes the first test is more convenient, sometimes the second.)

This theorem frequently enables us to replace a complicated infinitesimal, like $\pi(r+\Delta r)^2\Delta x$, by a simpler one, as $\pi r^2\Delta x$; *but it justifies this replacement only in the case expressly stated in the hypothesis of the theorem,* namely the case in which we are *finding the limit of a ratio.** (The fallacy that "infinitesimals of higher order can *always* be neglected" should be carefully guarded against.)

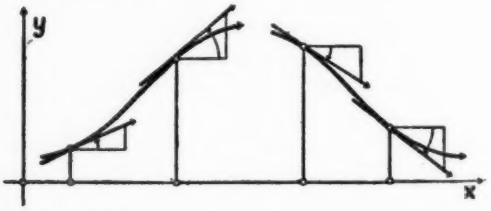
* A second replacement theorem for infinitesimals will be given in the chapter on Definite Integrals.

D. Applications of Differentiation in Studying the Properties of a Given Function.

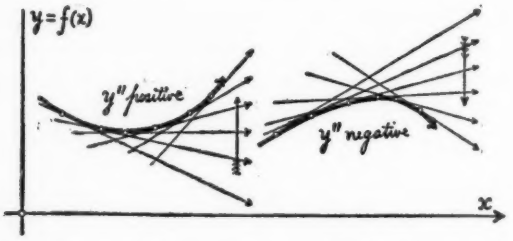
16. That a knowledge of differentiation is of fundamental importance in studying the variation of a given function is evident from the following theorems.

Let the given function be $y = f(x)$.

I. *The value of the derivative at any point shows the slope of the curve at that point.*



Hence, *if the derivative is positive* at any point, the curve is rising at that point (as we move in the positive direction along the axis); that is, *the function is increasing.* And *if the derivative is negative* at any point, the curve is falling at that point; that is, *the function is decreasing.*

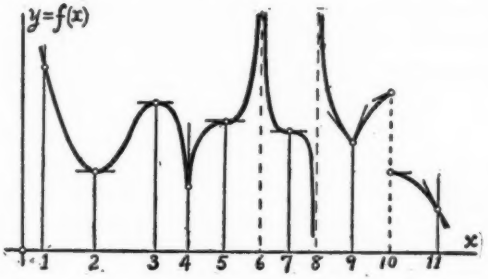


II. *If the second derivative is positive* at any point, the slope is increasing at that point, and hence *the curve is concave upward;* and *if the second derivative is negative* at any point, the slope is decreasing at that point, and hence *the curve is concave downward.*

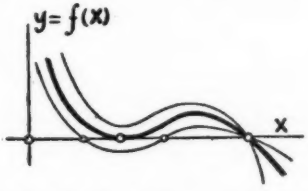
A point where the concavity changes sign is called a *point of inflexion;* at every such point, the second derivative is zero.*

17. Maxima and minima.—The application to problems in maxima and minima is immediate. In seeking the largest or smallest value of a given function in a given interval, we need consider only (1) the points where the slope is zero; (2) the points where the slope is infinite (or otherwise discontinuous); and (3) the end-points of the interval; for among these points the desired point will certainly be found. In most practical cases it will be a point where the slope is zero.

The conditions of the problem will usually show clearly which of these points, if any, is a maximum (or a minimum).



18. Multiple roots.—The roots, or the zeros, of a function, are the values of the argument for which the function becomes



zero. An inspection of the figure will show that any value of x for which $f(x)$ and $f'(x)$ are both zero simultaneously, will count as at least a double root.

* But the second derivative may be zero at points which are not points of inflexion; for example, $y = x^4$ at $x = 0$.

19. Small errors.—The following theorem is very useful in discussing the effect, on a computed value, of small errors in the data:

III. If dx is small, dy and Δy *are nearly equal.*

That is, the difference between dy and Δy can be made as small as we please, in comparison with dx, by making dx sufficiently small (except at points where dy/dx does not have a finite value).

Thus, if we wish to find approximately the error Δy produced by a small error in x, it will usually be sufficiently accurate to compute, instead of Δy, the simpler value, dy.

In problems concerning the *relative* error, dy/y, or dx/x, it is often convenient to take the logarithm of both sides of the given equation $y = f(x)$ before differentiating.

This class of problems is of great practical value.

CHAPTER III.

INTEGRATION AS THE INVERSE OF DIFFERENTIATION. SIMPLE DIFFERENTIAL EQUATIONS.

20. In many problems in pure and applied mathematics, we have given the derivative [or differential] of a function, and are required to find the function itself.

Suppose $f(x)$ [or $f(x)dx$] is the given derivative [or differential]; it is required to find a function $F(x)$ which, when differentiated, will give $f(x)$ [or $f(x)dx$]. Clearly, if one such function $F(x)$ has been found, then any function of the form $F(x)+C$, where C is any constant, will have the same property.

DEFINITION.—*Any function $F(x)$ whose differential is $f(x)dx$ is denoted by*

$$\int f(x)dx,$$

read: **an integral** *of $f(x)dx$.* The process of finding an integral of a function is called *integration,* or the *inverse of differentiation.*

If $F(x)$ is any particular integral of $f(x)dx$, then every integral of $f(x)dx$ can be expressed in the form $F(x)+C$, where C is a constant, called the *constant of integration.*

It can be shown that every continuous function has an integral; but this integral may not (in general, will not) be expressible in terms of the elementary functions.*

Most of the functions which occur in practice can, however, be integrated in terms of elementary functions, by the aid of a table of integrals, such as B. O. Peirce's well-known table of integrals. The entries in such a table can be verified by differentiation.

21. Formal work in integration.—The time devoted to the formal work of integration should not be longer than is nec-

* In such cases, an approximate expression for the integral may be obtained by infinite series.

essary to give the student a reasonable degree of expertness in the use of the tables.

The following integration formulas should be memorized; they are derived immediately from the corresponding formulas for differentiation.

$$\int cudx = c\int udx\,; \quad \int(u+v+\cdots)dx = \int udx + \int vdx + \cdots;$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} \text{ (provided } n \neq -1)\,;$$

(in words: an integral of any function raised to a constant power, $\neq -1$, times the differential of that function, is equal to the function raised to a power one greater, divided by the new exponent);

$$\int \frac{dx}{x} = \log_e x; \qquad \int e^x dx = e^x;$$

$$\int \sin x\,dx = -\cos x; \qquad \int \cos x\,dx = \sin x; \qquad \int \sec^2 x\,dx = \tan x;$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x, \text{ or } -\cos^{-1} x\,; \qquad \int \frac{dx}{1+x^2} = \tan^{-1} x.$$

The constant of integration must be supplied in each case.

A large number of integrals can be brought under the form $\int x^n dx$ by a simple transformation. For example,

$$\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx = \int (1-\sin^2 x)\cos x\,dx$$
$$= \int \cos x\,dx - \int \sin^2 x \cos x\,dx = \int \cos x\,dx - \int (\sin x)^2 d(\sin x)$$
$$= \sin x - (\sin x)^3/3.$$

Similarly for any odd power of the sine or cosine.

The following integrals are also important, though it is not worth while to memorize them when a table is at hand:

$$\int \sin^2 x dx = \tfrac{1}{2}(x - \sin x \cos x); \quad \int \cos^2 x dx = \tfrac{1}{2}(x + \sin x \cos x);$$

$$\int \frac{dx}{\cos x} = \log_e \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \tfrac{1}{2}\log_e \frac{1+\sin x}{1-\sin x}; \quad \int \frac{dx}{\sin x} = \log_e \tan \frac{x}{2};$$

$$\int \sinh x\,dx = \cosh x; \quad \int \cosh x\,dx = \sinh x; \quad \int \operatorname{sech}^2 x\,dx = \tanh x.$$

22. Among the other formulas of integration, the following are perhaps the ones that occur most often in practice; they are inserted here for reference, and especially to illustrate the usefulness of the hyperbolic functions.

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a},$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log_e \frac{a + x}{a - x} = \frac{1}{a} \tanh^{-1} \frac{x}{a},$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log_e \frac{x - a}{x + a} = -\frac{1}{a} \coth^{-1} \frac{x}{a},$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}, \quad \text{or} \quad = -\cos^{-1} \frac{x}{a},$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log_e (x + \sqrt{x^2 + a^2}), \quad \text{or} \quad = \sinh^{-1} \frac{x}{a},$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log_e (x + \sqrt{x^2 - a^2}), \quad \text{or} \quad = \cosh^{-1} \frac{x}{a},$$

$$\int \sqrt{a^2 - x^2}\, dx = \tfrac{1}{2}\left[x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a}\right],$$

$$\int \sqrt{x^2 + a^2}\, dx = \tfrac{1}{2}[x \sqrt{x^2 + a^2} + a^2 \log_e (x + \sqrt{x^2 + a^2})],$$

$$\text{or} = \frac{1}{2}\left[x \sqrt{x^2 + a^2} + a^2 \sinh^{-1} \frac{x}{a}\right],$$

$$\int \sqrt{x^2 - a^2}\, dx = \tfrac{1}{2}[x \sqrt{x^2 - a^2} - a^2 \log_e (x + \sqrt{x^2 - a^2})],$$

$$\text{or} = \frac{1}{2}\left[x \sqrt{x^2 - a^2} - a^2 \cosh^{-1} \frac{x}{a}\right].$$

23. Methods of Integration. Among the methods by which a given integral may be reduced to a form in the tables (or an integral in the table to one of the fundamental forms), the most important are (1) the method of substitution and (2) the method of integration by parts.

In the *method of substitution*, the given integral, $\int f(x)\, dx$, is expressed wholly in terms of some new variable y (and dy), in the hope that the new integral may be easier to handle than the old one. The substitutions which are most likely to be useful are the following:

(a) $y =$ any part of the given expression whose differential occurs as a factor; $y = x^k$; $y = 1/x$; $y = \sin x$; $y = \cos x$; $y = \tan(x/2)$.

(b) $x = a \sin y$, or $= a \tan y$, or $= a \sec y$, in expressions involving $\sqrt{a^2 - x^2}$, or $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$, respectively.

But much can be done without formal substitution of a new letter, if one remembers that the "x" in the formulas of integration may stand for any function.

The *method of integration by parts* is an application of the formula

$$\int u dv = uv - \int v du.$$

Take as dv a part of the given expression which can be readily integrated; on applying the formula, the new integral may be simpler than the old one.

The student should be practiced in both of these methods.

24. Simple differential equations. In a large number of problems in pure and applied mathematics, it is possible to write down an expression involving the rate of change of a desired function more readily than to write down the expression for the function itself. (Compare Chap. II, B.) In other words, it is often easier to write down a relation between the differentials of two variables than to write down the relation between the variables themselves. Such a relation connecting the differentials of two or more quantities, is called a *differential equation,* and any function which satisfies the equation, when substituted therein, is called a *solution* of the equation.

Every such problem, then, breaks up into two parts: (1) setting up the differential equation; (2) solving that equation.

The first part of the problem has already been treated in Chap. II, B. This part of the problem is too apt to be neglected in elementary courses; there is scarcely anything that develops real appreciation of the power of the calculus more effectively than practice in setting up for one's self the differential equations for various physical phenomena.

As to the second part of the problem, namely, the *solution* of the differential equation, the general plan is to reduce the given equation, by more or less ingenious devices, to the form $dy = f(x)\,dx$, or $y = \int f(x)\,dx$, and then to complete the solution, if possible, by the aid of a table of integrals. In a technical sense, the differential equation is said to be "solved" when it is thus reduced to a simple "quadrature"; that is, to a single integration.

The solution of a differential equation of the nth order, that is, an equation involving the nth derivative, will contain n arbitrary constants; to determine these constants, n conditions connecting x, y, y' . . ., $y^{(n)}$ must be known (the "initial" or "auxiliary" conditions of the problem).

25. The general discussion of differential equations is too large and too difficult a topic to find a place in a first course in the calculus, but two, at least, of the simpler equations are so important that their solution should be given, as an exercise in integration.

These equations are the following:

(1) $$\frac{dy'}{dt} + n^2 y = 0, \text{ where } y' = \frac{dy}{dt}.$$

The solution is

$$y = C_1 \sin(nt + C_2) \text{ or, } y = C_3 \sin nt + C_4 \cos nt,$$

where the C's are arbitrary constants.

(2) $$\frac{dy'}{dt} - n^2 y = 0, \text{ where } y' = \frac{dy}{dt}.$$

The solution is

$$y = C_1 \sinh(nt + C_2), \text{ or, } y = C_3 e^{nt} + C_4 e^{-nt},$$

where the C's are arbitrary constants.

The method of obtaining these results, rather than the results themselves, should be remembered: namely, multiply through by dy, noting that $dy/dt = y'$ and integrate each term, getting $\frac{1}{2}y'^2 + \frac{1}{2}n^2y^2 = C$; then replace y' by dy/dt, "separate

the variables," and integrate again. By a similar method, any equation of the form $dy'/dt + f(y) = 0$ can be solved, if we can integrate $f(y)\,dy$.

26. Another very important differential equation is the equation for "damped vibration":

$$\frac{dy'}{dt} + 2b\frac{dy}{dt} + a^2y = 0, \text{ where } y' = \frac{dy}{dt}.$$

The solution is given here for reference:

Case 1. If $a^2 - b^2 > 0$, let $m = \sqrt{a^2 - b^2}$; then

$$y = C_1e^{-bt}\sin(mt + C_2),$$

$$\text{or } y = [C_3\sin(mt) + C_4\cos(mt)]e^{-bt}.$$

Case 2. If $a^2 - b^2 = 0$,

$$y = e^{-bt}(C_1 + C_2t).$$

Case 3. If $a^2 - b^2 < 0$, let $n = \sqrt{b^2 - a^2}$; then

$$y = C_1e^{-bt}\sinh(nt + C_2),$$

$$\text{or } y = C_3e^{-(b+n)t} + C_4e^{-(b-n)t}.$$

CHAPTER IV.

INTEGRATION AS THE LIMIT OF A SUM. DEFINITE INTEGRALS.

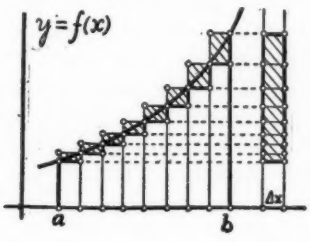
27. The limit of a sum. Many problems in pure and applied mathematics can be brought under the following general form:

Given, a continuous function, $y=f(x)$, from $x=a$ to $x=b$. Divide the interval from $x=a$ to $x=b$ into n equal parts, of length $\Delta x=(b-a)/n$. Let $x_1, x_2, x_3, \ldots x_n$ be values of x, one in each interval; take the value of the function at each of these points, and multiply by Δx; then form the sum:*

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

Required, the limit of this sum, as n increases indefinitely, and $\Delta x = 0$.

This problem may be interpreted geometrically as the problem of finding the area under the curve $y=f(x)$, between the ordinates $x=a$ and $x=b$; each term of the sum represents



the area of a rectangle whose base is Δx and whose altitude is the height of the curve at one of the points selected. It is easily seen that the difference between the sum of the rectangles and the area of the curve is less than a rectangle

* It is not necessary that the parts be equal, provided the largest of them approaches zero when n is made to increase indefinitely.

whose base is Δx and whose altitude is constant. This difference approaches zero as $\Delta x \doteq 0$; therefore the sum of the rectangles approaches the area of the curve as a limit.

In this way, or by an analytic proof, it is shown that the limit of the sum in question always exists. The problem then is, to find the value of this limit.

The value of the limit can always be obtained by the following fundamental theorem, whenever an integral of the given function $f(x)$ can be found.

FUNDAMENTAL THEOREM OF SUMMATION. *If $x_1, x_2, \cdots x_n$ are values of x ranging from $x = a$ to $x = b$, as in the statement of the general problem above, then*

$$\lim_{\Delta x \doteq 0} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x] = F(b) - F(a),$$

where

$$F(x) = \int f(x)dx$$

is any function whose derivative is the given function $f(x)$.

The proof of this remarkable theorem is best given by showing that the right hand side of the equation, as well as the left, is equal to the area under the curve from $x = a$ to $x = b$; to do this, consider the area from $x = a$ to a variable point $x = x$, and find the rate of change of this area regarded as a function of x; hence find the area itself as a function of x, determine the constant of integration in the usual way, and then put $x = b$ in the result.

DEFINITION. *The limit of a sum of the kind described above is called the* **definite integral** *of $f(x)dx$ from $x = a$ to $x = b$, and is denoted by*

$$\lim_{n \doteq \infty} \sum_{i=0}^{i=n} f(x_i)\Delta x, \quad \text{or} \int_{x=a}^{x=b} f(x)dx.$$

The function obtained by the inverse of differentiation is called, for distinction, an *indefinite* integral. By the fundamental theorem just stated, the definite integral is equal to the difference between two values of the indefinite integral:

$$\int_{x=a}^{x=b} f(x)dx = \left[\int f(x)dx\right]_{x=b} - \left[\int f(x)dx\right]_{x=a}.$$

The double use of the term "integration"—meaning in one case anti-differentiation, and in the other case finding the limit of a sum—and the fundamental theorem connecting these two distinct concepts, should be made thoroughly clear.*

The concept of the definite integral is the most useful concept in the application of the calculus, and the study of problems which can be formulated as definite integrals may well occupy one third of the time of a first course.

For example, problems in areas, volumes, surfaces, length of arc, center of gravity, moments of inertia, center of fluid pressure, etc. *Many of these problems require two applications of the fundamental theorem.*

28. Properties of definite integrals. From the definition of the definite integral we have at once:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx;$$

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx;$$

and, by the aid of a figure, the *Mean Value theorem:*

$$\int_a^b F(x)f(x)dx = F(X)\int_a^b f(x)dx,$$

where X is some (unknown) value of x between a and b, and $F(x)$ is any continuous function which *does not change sign* from $x=a$ to $x=b$.

We have also the following important theorem on *change of variable:*

In evaluating the integral

$$\int_{x=a}^{x=b} f(x)dx,$$

if x is a function of a new variable t, we may replace $f(x)dx$ by its value in terms of t and dt, and replace $x=a$ and $x=b$

* The use of the term in the sense of summation was historically the earlier, and the symbol $\int$ is the old English "long s," the first letter of "sum."

by the corresponding values $t = a$ and $t = \beta$, without altering the value of the integral, provided that thoroughout the interval considered there is one and only one value of x for every value of t, and one and only one value of t for every value of x.

29. All problems leading to a definite integral are problems in finding the limit of a sum, each term of which is approaching zero, while the number of terms is increasing indefinitely. Whenever a function $f(x)$ can be found, such that all terms of the sum are obtained by substituting successively x_1, x_2, etc., in the expression $f(x)dx$, then the formulation of the problem as a definite integral is immediately obvious. The separate terms of the sum, of which $f(x_k)dx$ is a type, are called *elements*.

Thus, in finding the area under a curve, an obvious element of area is the rectangle ydx; if the curve revolves about the x-axis, the element of volume of the solid thus generated is the cylinder $\pi y^2 dx$. Here y must be expressed as a function of x before the integration can be completed. Again, in polar coördinates, the element of area is the sector, $\frac{1}{2}r^2 d\theta$, where r must be a known function of θ.

35. In many cases, however, the proper function is not so immediately obvious. In such cases, the following theorem is of great service:

SECOND REPLACEMENT THEOREM FOR INFINITESMALS (THEOREM OF DUHAMEL). *In finding the limit of a sum of positive terms, each of which is less than a quantity which approaches zero while the number of terms increases indefinitely, any term may be replaced by a "similar" term without affecting the value of the limit. Two variables α and β are called "similar" if*

$$(1) \qquad \lim \frac{\alpha}{\beta} = 1, \quad \textit{or} \quad \textit{if} \ (2) \ \lim \frac{\alpha - \beta}{\alpha} = 0.$$

For example, let us find the weight of a rod whose density, w, and cross-section, A, are both functions of x. The "true element" of weight, ΔW, corresponding to a given length Δx, will certainly lie between the values $w'A'\Delta x$ and $w''A''\Delta x$, where

w', A' are the smallest values, and w'', A'' the largest values of w and A within the interval from $x = x$ to $x = x + \Delta x$; but either of these extreme values may be replaced by the simpler value $wA\Delta x$, where w, A are the values of w and A at the *beginning* of the interval, for,

$$\lim \frac{w'A'\Delta x}{wA\Delta x} = \lim \frac{w''A''\Delta x}{wA\Delta x} = 1.$$

Hence, ΔW itself, which lies between these extremes, can be replaced by $wA\Delta x$, which is therefore the required "differential element" of weight.* The total weight of the rod, from $x = a$ to $x = b$, is then equal to the definite integral

$$\int_{x=a}^{x=b} wA\,dx;$$

where w and A must of course be expressed as functions of x before the integration can be completed.

In justifying replacements of this kind by Duhamel's theorem, sometimes the first test is more convenient, sometimes the second. When once the common replacements have been justified, the use of the theorem in practice rapidly becomes almost intuitive.

30. Approximate methods of integration.—If the function $f(x)$ is given only empirically, the theorem on evaluating the definite integral by purely mathematical means cannot be applied. In such cases, an approximate value of the definite integral

$$\int_{x=a}^{x=b} f(x)\,dx$$

may be found by plotting the curve $y = f(x)$ on squared paper, and estimating the area by *counting squares* (and fractions of squares).

Another method of approximation is by *Simpson's Rule:*

* When x is the independent variable, it is immaterial whether we write Δx or dx.

Divide the area into n panels, where n is *even*, and number the ordinates from 1 to $n+1$; then, if Δx is the width of each panel,

$$\text{Area} = \tfrac{1}{3}\Delta x \text{ (first ordinate + last ordinate}$$
$$\text{+ twice the sum of the other odd ordinates}$$
$$\text{+ four times the sum of the even ordinates).}$$

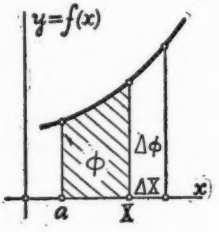
The instrument known as a *planimeter* provides a mechanical means of integration, used especially in measuring the areas of indicator cards.

Another and very important method of approximation is by the use of *series;* see the next chapter.

31. Definite Integral as a function of its upper limit.—If X is a variable, the definite integral

$$\int_{x=a}^{x=X} f(x)dx$$

represents the area under the curve $y=f(x)$ from $x=a$ to the variable ordinate $x=X$, and is therefore a function of X,



say $\phi(X)$. By applying the definition of derivative to this function, it is easy to see from the figure that $\phi'(X)=f(X)$:

$$\frac{d}{dX}\int_{x=a}^{x=X} f(x)dx = f(X).$$

Thus $\phi(X)$ is one of the indefinite integrals of $f(X)$.

Any indefinite integral which cannot be expressed in terms of known functions can always be written as a definite integral regarded as a function of its upper limit, and its value, for any given value of the argument, can then be found by one of the methods of approximate integration.

The elliptic integrals, the most important of which are

$$\int_{\theta=0}^{\theta=\phi} \frac{d\theta}{\sqrt{1-(k^2)\sin^2\theta}} \quad \text{and} \quad \int_{\theta=0}^{\theta=\phi} \sqrt{1-(k^2)\sin^2\theta}\, d\theta,$$

are handled in this way, by the method of expansion in series. The student should be made familiar with the construction and use of tables of the elliptic integrals.

In such tables, k^2 is usually expressed in the form $\sin^2\alpha$, which emphasizes the fact that $k^2 \leq 1$.

CHAPTER V.

APPLICATIONS TO ALGEBRA: EXPANSION IN SERIES; INDETERMINATE FORMS.

Note.—This chapter may be taken, if preferred, immediately after the chapter on differentiation. It is in reality an extension of the "formal work" of that chapter, since it deals with changes in the *form* of algebraic expressions.

32. Taylor's theorem.—It is often desirable to obtain an approximate expression for a given function, in the neighborhood of a given point $x = a$, in the form of a series arranged according to ascending powers of $x - a$, with constant coefficients. For values of x near to a, the higher powers of $x - a$ will then become negligible.

The most convenient theorem for this purpose is the following:

TAYLOR'S THEOREM. *If $f(x)$ is continuous, and has derivatives through the $(n+1)$st, in the neighborhood of a given point $x = a$, then, for any value of x in this neighborhood,*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \frac{f^{n+1}(X)}{(n+1)!}(x-a)^{n+1},$$

where X is some unknown quantity between a and x. The last term,

$$R = \frac{f^{n+1}(X)}{(n+1)!}(x-a)^{n+1},$$

is the *error* committed if we stop the series with the term in $(x-a)^n$, and the formula is useful only when this error becomes smaller and smaller as we increase the number of terms.

This form for the "remainder," R is easily remembered since it differs from the general term of the series only by the fact that the derivative in the coefficient of the power of $(x-a)$ is taken for $x=X$ instead of for $x=a$.* (There are also other forms of the remainder which are sometimes useful.)

33. The special case where $a=0$ is called **Maclaurin's Theorem:**

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^n(0)}{n!}x^n+\frac{f^{n+1}(X)}{(n+1)!}x^{n+1},$$

where X is some unknown quantity between 0 and x.

34. Another special case, obtained by putting $n=0$, gives

$$f(x)-f(a)=f'(X)(x-a),$$

where again X is some unknown quantity between a and x. This theorem is called the **Law of the Mean,** and is of great importance in the theoretical development of the subject.

35. If the error-term in Taylor's Theorem approaches zero as n increases, the formula becomes a convergent infinite series, called the **Taylor's series** for the given function, about the given point $x=a$.

The series with which the student should be especially familiar are the following:

---

* The simplest proof of this theorem is by means of integration. For example, for the case $n=2$, we have

$$\int_a^x f'''(t)dt=f'''(X)(x-a),$$

where X is some (unknown) constant between a and x (as is evident from a figure); but also

$$\int_a^x f'''(t)dt=f''(x)-f''(a),$$

by the fundamental theorem; so that

$$f''(x)-f''(a)=f'''(X)(x-a).$$

Integrating this equation twice between the limits $x=a$ and $x=x$, remembering that $f''(a)$ and $f'''(X)$ are constants, we have at once:

$$f'(x)-f'(a)-f''(a)(x-a)=f'''(X)\tfrac{1}{2}(x-a)^2,$$
$$f(x)-f(a)-f'(a)(x-a)-f''(a)\tfrac{1}{2}(x-a)^2=f'''(X)\tfrac{1}{2}\tfrac{1}{3}(x-a)^3.$$

Binomial series:

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots,$$

provided $|x| < 1$.

Sine series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad (x \text{ in radians}).$$

Cosine series:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad (x \text{ in radians}).$$

Exponential series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

Next in importance are the series for $\log(1+x)$, $\tan^{-1} x$, $\sinh x$, and $\cosh x$.

From these series we have the following important approximations, when x is small:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \cdots, \quad \frac{1}{1+x} = 1 - x + \cdots, \quad \frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \cdots,$$

$$\sin x = x - \cdots, \quad \cos x = 1 - \cdots, \text{ etc.}$$

An important special case of the binomial series is the geometric series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \quad \text{provided } |x| < 1.$$

36. The student should also understand the comparison test, and the test-ratio test, for the convergence of an infinite series, and the following theorem on *alternating* series: If the terms of a series are alternately positive and negative, each being numerically less than or equal to the preceding, and if the nth term approaches zero as n increases, then the series is convergent, and the error made by breaking off the series at any given term does not exceed numerically the value of the last term retained.

Further, *a power series can be differentiated or integrated term by term, within the interval of convergence.*

37. Indeterminate forms.—The evaluation of indeterminate forms can often be facilitated by the use of the following theorem, in which $f(x)$ and $F(x)$ are functions which possess derivatives at a given point $x=a$.

Theorem of indeterminate forms. If $f(x)$ and $F(x)$ both approach zero, or both become infinite, when x approaches a, then

$$\lim_{x=a}\left[\frac{f(x)}{F(x)}\right]=\lim_{x=a}\left[\frac{f'(x)}{F'(x)}\right].$$

The second limit may often be easier to evaluate than the first.

The student should thoroughly understand the meaning of indeterminate forms, for which the common symbols $\frac{0}{0}$, 1^{∞}, etc., are merely a suggestive short-hand notation.

Thus, "0/0" means that we are asked to find the limit of a function $y=f(x)/F(x)$, when $f(x)$ and $F(x)$ both approach zero. Now the change in $f(x)$ alone would tend to decrease y numerically, while the change in $F(x)$ alone would tend to increase y; hence we cannot tell, without further investigation, what the combined effect of both changes, taking place simultaneously, will be.

Again, the symbol 1^{∞} means that we are asked to find the limit of a function $y=f(x)^{F(x)}$, when $f(x)$ approaches 1 and $F(x)$ becomes infinite. Now the change in $f(x)$ alone would tend to make y approach 1, while the change in $F(x)$ alone would tend to make y recede from 1; hence we cannot tell, without further investigation, what the combined effect will be.

The student should thoroughly master in this way the meaning of all the seven types of indeterminate forms, namely,

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ 0^0,\ 1^{\infty},\ \infty^0,\ \infty-\infty.$$

The cases involving exponents are best treated by first finding the limit of the logarithm of y, from which the limit of y

can then be obtained. The form $0 \cdot \infty$, or $y = f(x) \cdot F(x)$, can be written as $y = \dfrac{f(x)}{1/F(x)}$, or $y = \dfrac{F(x)}{1/f(x)}$, which then comes under one of the first two forms. The last form, $\infty - \infty$, is usually best handled by the method of series.

Before applying the theorem of indeterminate forms, one should, of course, try first to find the required limit by a simple algebraic transformation, if possible.

CHAPTER VI.

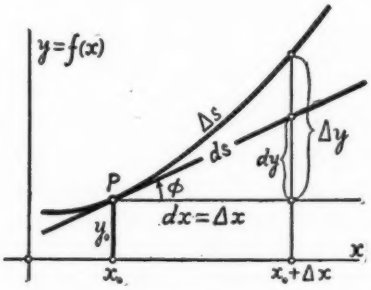
APPLICATIONS TO GEOMETRY AND MECHANICS.

In all applications to geometry, in which a curve is represented by an equation connecting x and y, the *scales on the x and y axes must be equal* (compare §3, footnote).

38. Tangent and normal.—The equation of the tangent at any point (and hence the equation of the normal) can be written down at once when we know the slope and the coördinates of the point of contact.

Again, to find the *subtangent* or *subnormal* at any point, we have simply to find the ordinate and the slope at that point, and then solve a right triangle.

39. Differential of arc. If s = length of arc of the curve $y=f(x)$, measured from some fixed point A of the curve, then s, like y, is a function of x, and we may ask what is the



rate of change of s with respect to x, that is, what is the value of ds/dx. Now $ds/dx = \lim\ (\Delta s/\Delta x)$, and in finding this limit we may replace the arc Δs by its chord, $\sqrt{(\Delta x)^2+(\Delta y)^2}$; hence $ds/dx = \lim \sqrt{1+(\Delta y/\Delta x)^2} = \sqrt{1+(dy/dx)^2}$, or

$$ds = \sqrt{(dx)^2+(dy)^2},$$

as indicated in the figure. This formula, and the corresponding relations

$$dx = ds \cos \phi, \quad dy = ds \sin \phi,$$

are important, and are readily recalled to mind by the figure.

In the case of a circle of radius r, if $d\theta$ = the angle at the center, subtended by the arc ds, then

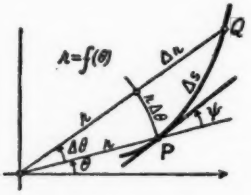
$$ds = r d\theta,$$

provided the angle is measured in radians.

40. Again, in case of a curve whose equation is given in polar coordinates, $r = f(\theta)$, we see at once from the figure, by the aid of the replacement theorem, that

$$ds = \sqrt{(dr)^2 + (rd\theta)^2} \quad \text{and} \quad \tan \psi = \frac{rd\theta}{dr},$$

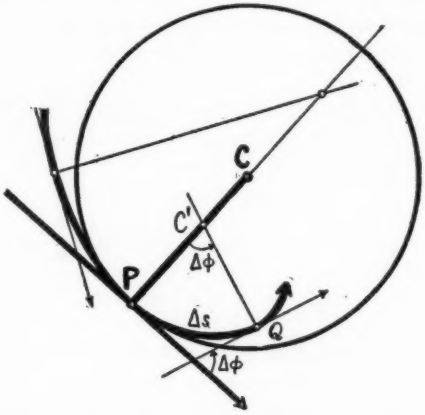
where ψ is the angle which the tangent makes with the radius vector produced.



41. Radius of Curvature.—Consider the normal to a given curve at a given point, P, and also the normal at a neighboring point, Q. These two normals will intersect at some point C' on the concave side of the curve; and as Q approaches P, along the curve, this point C' will (in general) approach a definite position C as a limit. The circle described with a center at this point C and radius equal to CP will fit the given curve more closely, in the neighborhood of the point P, than does any other circle. This circle is called the *osculating circle,* or the circle of curvature, at the point P; its center C is called the *center of curvature,* and its radius CP is called the *radius of curvature,* at the point P.

The radius of curvature may thus be taken as a measure of the flatness or sharpness of the curve; the smaller the radius of curvature, the sharper the curve.

The length of the radius of curvature, R, at any point P is most readily found as follows: In the triangle $PC'Q$, we have $C'P/PQ = \sin Q/\sin \Delta\phi$, where $\Delta\phi$ is the angle between the normals (or between the tangents) at P and Q. Therefore $R = \lim C'P = \lim$ (chord $PQ/\sin \Delta\phi$) $\sin Q$; or, replacing



the chord by the arc Δs, and $\sin \Delta\phi$ by $\Delta\phi$, and noticing that Q is approaching 90°, so that $\lim \sin Q = 1$, we have $R = \lim (\Delta s/\Delta\phi)$, or,

$$R = \frac{ds}{d\phi}.$$

This important formula is readily recalled to mind from the figure, if one thinks of the arc Δs as approximately a circular arc.

To express R in terms of x and y, we have only to remember that $ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + y'^2}dx$, and $\tan \phi = dy/dx = y'$, whence $d\phi = y''dx/(1 + y'^2)$; then

$$R = \frac{(1 + y'^2)}{y''}$$

Def. The *curvature* of a curve at a curve is defined as the rate at which the angle ϕ is changing with respect to the length of arc s; that is,

$$\text{curvature} = \frac{d\phi}{ds} = \frac{1}{R}.$$

If the slope of the curve is small, the curvature is approximately equal to y''.

Def. The locus of the center of curvature is called the *evolute* of the curve.

The normals to the given curve are tangent to the evolute, and the given curve may be traced by unwinding a string from the evolute.

42. Velocity and acceleration.—Consider a particle moving along a *straight line.* Its distance from the origin is a function of the time:

$$x = F(t).$$

The *velocity* of the particle is the *rate of change of its distance:*

$$v = dx/dt = F'(t) = x'.$$

The velocity will be positive or negative, according as the particle is moving forward or backward along the line.

The *acceleration* of the particle is the *rate of change of its velocity:*

$$A = dv/dt = F''(t) = x''.$$

The acceleration will be positive or negative according as the velocity is increasing or decreasing (algebraically).

If a particle is moving along a *plane curve,* we must consider the *components* of its motion along two fixed axes. The components of acceleration along the x- and y-axes are x'' and y''; the components of acceleration along the tangent and normal are dv/dt and v^2/R, respectively, where $v = \sqrt{x'^2 + y'^2}$ = the path velocity, and R = the radius of curvature.

It should be carefully noticed that dv/dt is not the whole acceleration, but only that component of the acceleration which lies along the tangent.

The importance of this application in problems in mechanics is obvious.

Note.—As explained in the preface of this report, these pages are intended merely to give a résumé of the working principles of the calculus with which the student should be perfectly familiar after having taken a course in this subject. *The main part of the work of such a course should be problems done by the students—each problem being solved on the basis of the small number of fundamental theorems here mentioned.*